Fundamental matrix computation and triangulation

December 14, 2021

Triangulation

- We have $\mathrm{x} = P\mathrm{X}$ and $\mathrm{x}' = P'\mathrm{X}$.
- Applying vector product by x on the first equation gives

$$
\begin{aligned}
x(\mathrm{p}^{3T}\mathrm{X}) - (\mathrm{p}^{1T}\mathrm{X}) &= 0 \\
y(\mathrm{p}^{3T}\mathrm{X}) - (\mathrm{p}^{2T}\mathrm{X}) &= 0 \\
x(\mathrm{p}^{2T}\mathrm{X}) - (\mathrm{p}^{1T}\mathrm{X}) &= 0
\end{aligned}
$$

- Last equation is linearly dependent.
- Similarly we get set of linear equations from the second equation.

Triangulation

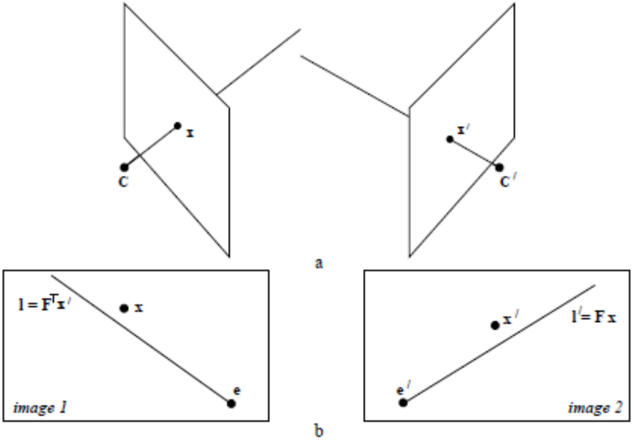
- These four equations can be rewritten in matrix form $AX = 0$ with

$$A = \begin{pmatrix} x\mathrm{p}^{3T} - \mathrm{p}^{1T} \\ y\mathrm{p}^{3T} - \mathrm{p}^{2T} \\ x'\mathrm{p}'^{3T} - \mathrm{p}'^{1T} \\ y'\mathrm{p}'^{3T} - \mathrm{p}'^{2T} \end{pmatrix}$$

  p^{iT} is i-th row of P. We seek solution that minimizes $\|AX\|$ with $\|\mathrm{X}\| = 1$.
- This solution is not geometrically optimal, the image points do not satisfy epipolar constraint $\mathrm{x}'^T F\mathrm{x}$

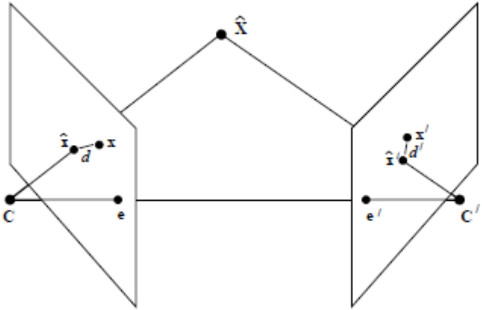
Triangulation

- Back projections of x′ and x do not meet.
- We want a method that is projectively invariant. If H is projective transformation of space and τ is triangulation method we have
$$X = \tau(x, x', P, P')$$
and
$$\tau(x, x', P, P') = H^{-1}\tau(x, x', HP, HP')$$


x
C
x′
C′
a
l = F^T x′
x
e
image 1
x′
l′= F x
e′
image 2
b

Triangulation

- For a measured pair $x \leftrightarrow x'$ we seek a pair $\hat{x} \leftrightarrow \hat{x}'$ such that $\hat{x}' F \hat{x} = 0$ and
$$d(x, \hat{x})^2 + d(x', \hat{x}')$$
is minimal.



$\hat{X}$
$\hat{x}$
x
d
$\hat{x}'$
x'
d'
C
e
e'
C'

- A pair of points $\hat{x} \leftrightarrow \hat{x}'$ that satisfy epipolar constraint lies on corresponding epipolar lines.
- Points on the epipolar lines that are closest to x and x' are orthogonal projections of x and x' onto the epipolar lines.
- We seek to minimize

$$d(x, l)^2 + d(x', l')$$

Strategy

- Parametrize pencil of epipolar lines $l(t)$ in the first image by a parameter t.
- Using F compute the pencil $l'(t)$.
- Express the distance function $d(x, l(t))^2 + d(x', l'(t))$ explicitly as a function of t.
- Find value of t that minimizes the distance function

Algorithm

- Translate coordinates in the first and second images such that the points x and x' have coordinates $(0, 0, 1)$.
- Rotate coordinates about $(0, 0, 1)$ in a way that epipoles e and e' have coordinates $(1, 0, f)$ and $(1, 0, f')$.
- Transform the fundamental matrix using the above transformations
- Because $e'^T F = 0$ and $Fe = 0$ the fundamental matrix can be then written as
$$\begin{pmatrix} ff'd & -f'c & -f'd \\ -fb & a & b \\ -fd & c & a \end{pmatrix}$$
- $d(x, l(t))^2 = \frac{t^2}{1+(tf)^2}$, $d(x,' l(t))^2 = \frac{(ct+d)^2}{(at+b)^2+(f'(ct+d))^2}$

Algorithm

- Compute total squared distance $s(t) = d(x, l(t))^2 + d(x,' l(t))^2$ and its derivation $s'(t) = \frac{g(t)}{h(t)}$
- Find roots of $g(t)$.
- Take a root t_{min} of $g(t)$ such that the value $s(t_{min})$ is global minimum of s and find asymptotical value $s(\inf)$. If $s(\inf)$ is global minimum, set $t_{min} = \infty$ (as a limit).
- Evaluate $l = (tf, 1, -t)$ and l' at t_{min}
- Find closest points $\hat{x}$ and $\hat{x}'$ on these lines to the origin.
- Transform computed points back to the original coordinates.
- Use linear triangulation for F, $\hat{x}$ and $\hat{x}'$.