Fundamental matrix computation and triangulation

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Basic equations

For any pair of matching points $\mathrm{x} \leftrightarrow \mathrm{x}'$ the fundamental matrix is defined by

$$\mathrm{x}'F\mathrm{x} = 0$$

- Rank $F = 2$,
- F has 7 DOF

Basic equations

In coordinates $\mathrm{x} = (x, y, 1)^T$, $\mathrm{x}' = (x', y', 1)^T$,

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

$$\mathrm{f} = (f_{11}, f_{12}, \ldots, f_{33})^T$$

- Pair $\mathrm{x} \leftrightarrow \mathrm{x}'$ is associated with equation

$$(x'x, x'y, x', y'x, y'y, y', x, y, 1)\mathrm{f} = 0$$

Basic equations

For n pairs we get system of linear equations

$$Af = \begin{pmatrix} x_1'x_1 & x_1'y_1 & x_1' & y_1'x_1 & y_1'y_1 & y_1' & x_1 & y_1 & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ x_n'x_n & x_n'y_n & x_n' & y_n'x_n & y_n'y_n & y_n' & x_n & y_n & 1 \end{pmatrix} f = 0$$

- Solution f is determined up to scale.
- Solvabililty condition $\mathrm{rank}(A) \leq 8$
- If $\mathrm{rank}(A) = 8$ then solution is unique up to scale.

Basic equations

In case $\mathrm{rank}(A) = 9$ (imperfect pairing) we seek a solution with LSM

$$Af = 0 \qquad\qquad ||f|| = 1$$

- f is the singular vector associated with the smallest singular value of A.
- If $A = UDV^T$ is the singular decomposition of A, then f is the last column of V.

Basic equations

Problem: The matrix F that solves $Af = 0$ need not be of $\mathrm{rank}(F) = 2$
Consequence: Epipolar lines do not intersect at unique point.
Solution: Find matrix F' close to F with $\mathrm{rank}(F') = 2$.

- "Closeness" is measured by Frobenius norm.
- We seek F' such that $\mathrm{rank}(F') = 2$ and $\|F - F'\|$ is minimal.

We use SVD

- $F = U \,\mathrm{diag}(r, s, t)\, V^T$, $r \geq s \geq t$.
- $F' = U \,\mathrm{diag}\,(r, s, 0) V^T$.

8-point algorithm

Task: Find F from correspondences $\mathrm{x} \leftrightarrow \mathrm{x}'$

- Linear solution of $A\mathrm{f} = 0$
- Forcing regularity constrain.

In this form the algorithm is badly conditioned. For better stability a normalization process is needed:

- Transform points $\{\mathrm{x}_i\}$ and $\{\mathrm{x}'_i\}$ using translation and scaling transformations such that their centroid become $(0, 0)$ and mean distance from centroid will be $\sqrt{2}$.
- This will give transformations T for $\{\mathrm{x_i}\}$ and T' for $\{\mathrm{x_i}'\}$

8-point algorithm

Task: Find F from correspondences $\mathrm{x} \leftrightarrow \mathrm{x}'$

- Normalize points $\{\mathrm{x_i}\}$ and $\{\mathrm{x_i}'\}$ using transforms T and T'.

$$\hat{\mathrm{x}}_\mathrm{i} = T\mathrm{x_i}$$
$$\hat{\mathrm{x}}'_\mathrm{i} = T'\mathrm{x_i}'$$

- Linear solution of $\hat{A}\hat{\mathrm{f}} = 0$ gives $\hat{F}$.
- Force regularity constrain for $\hat{F}$.
- Denormalisation: $F = {T'}^T \hat{F} T$.
- There are further numerical methods to improve the solution

Geometric error

Given set of correspondences of points between two images, find homography H that transform one image to the other.

- $\tilde{\mathrm{x}}$ – true point
- x – measured point
- $\hat{\mathrm{x}}$ – estimated point

Transfer error

$$\sum_i d(\mathrm{x_i}', H\tilde{\mathrm{x}}_i)^2$$

$\hat{H}$ is minimizer of transfer error. It is the sought–after solution.

Geometric error – Transfer error

Transfer error

$$\sum_i d(\mathrm{x_i}', H\tilde{\mathrm{x}}_i)^2$$

$\hat{H}$ is minimizer of transfer error. It is the sought–after solution.

Geometric error – Symmetric transfer error

Transfer error

$$\sum_i d(\mathrm{x_i}', H\tilde{\mathrm{x}}_i)^2 + d(\mathrm{x_i}, H^{-1}\tilde{\mathrm{x}}_i')^2$$

$\hat{H}$ is minimizer of transfer error. It is the sought–after solution.

Geometric error – Reprojection error

Find $\hat{H}$ and $\hat{x}_i$ and $\hat{x}_i' = \hat{H}\hat{x}_i$ such that

$$\sum_i d(x_i', \hat{x}_i)^2 + d(x_i', \hat{x}_i')^2$$

is minimal.

Gold Standard algorithm

1. First estimate $\hat{F}$ using 8-point algorithm
2. Estimates for $\hat{\mathrm{x}}_{\mathrm{i}}$ and $\hat{\mathrm{x}}'_{\mathrm{i}}$:
   2.1 Chose $P = [I|0]$, $P' = [[\mathrm{e}']_\times \hat{F}|e']$
   2.2 Pair $\mathrm{x_i} \leftrightarrow \mathrm{x}'_{\mathrm{i}}$ and $\hat{F}$ gives estimate for $\hat{\mathrm{X}}_i$ using triangulation.
   2.3 $\hat{\mathrm{x}}_i = P\hat{X}_i$ and $\hat{\mathrm{x}}'_i = P'\hat{X}_i$
3. Minimize

$$\sum_i d(\mathrm{x_i}', \hat{\mathrm{x}}_i)^2 + d(\mathrm{x_i}', \hat{\mathrm{x}}'_i)^2$$

over $\hat{F}$ and $\hat{X}_i$. (There is a numerical method – Levenberg–Marquardt algorithms for this step.)

Triangulation

- We have $\mathrm{x} = P\mathrm{X}$ and $\mathrm{x}' = P'\mathrm{X}$.
- Applying vector product by x on the first equation gives

$$
\begin{aligned}
x(\mathrm{p}^{3T}\mathrm{X}) - (\mathrm{p}^{1T}\mathrm{X}) &= 0 \\
y(\mathrm{p}^{3T}\mathrm{X}) - (\mathrm{p}^{2T}\mathrm{X}) &= 0 \\
x(\mathrm{p}^{2T}\mathrm{X}) - (\mathrm{p}^{1T}\mathrm{X}) &= 0
\end{aligned}
$$

- Last equation is linearly dependent.
- Similarly we get set of linear equations from the second equation.

Triangulation

- These four equations can be rewritten in matrix form $AX = 0$ with

$$A = \begin{pmatrix} x\mathrm{p}^{3T} - \mathrm{p}^{1T} \\ y\mathrm{p}^{3T} - \mathrm{p}^{2T} \\ x'\mathrm{p}'^{3T} - \mathrm{p}'^{1T} \\ y'\mathrm{p}'^{3T} - \mathrm{p}'^{2T} \end{pmatrix}$$

  p^{iT} is i-th row of P.
  We seek solution that minimizes $\|A\mathrm{X}\|$ with $\|\mathrm{X}\| = 1$.