Epipolar Geometry

Epipolar geometry

- Images of a scene obtained by two cameras

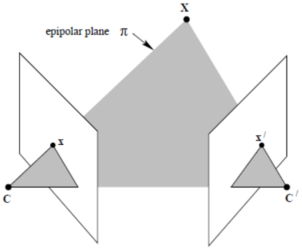

Figure: R. Hartley, A. Zisserman: Multiple View Geometry in Computer Vision

- How point $\mathbf{x}$ constrains point $\mathbf{x}'$.
- How to find camera matrices P and P' from point correspondences $\mathbf{x} \leftrightarrow \mathbf{x}'$.
- How to find point $\mathbf{X}$ corresponding to a pair $\mathbf{x} \leftrightarrow \mathbf{x}'$ if we know P and P'.

Epipolar geometry – Definitions

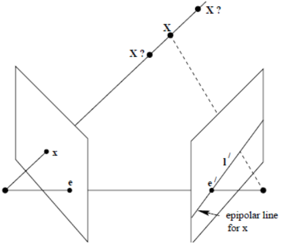

Figure: R. Hartley, A. Zisserman: Multiple View Geometry in Computer Vision

- $\mathbf{C}$, $\mathbf{C}'$ – Camera centers
- π – Epipolar plane. Contains $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{X}$ and its images $\mathbf{x}$ and $\mathbf{x}'$.
- Baseline – The line joining camera centres; it is the intersection of all epipolar planes.

Epipolar geometry – Definitions

- Epipoles – Intersection of baseline with image planes.
- Epipolar line – Intersection of an epipolar plane with image plane of a camera. Point $\mathbf{x}$ in image plane of first camera determines a ray through $\mathbf{C}$ and $\mathbf{x}$. Its image in image plane of the second camera is epipolar line $\mathbf{l}'$ for $\mathbf{x}$
- Epipole lies in intersection of all epipolar lines.

Fundamental matrix

- We have mapping from image plane of first camera to the space of lines through epipole in the image plane of the second camera defined as

$$\mathbf{x} \mapsto \mathbf{l}',$$

i.e. it maps point $\mathbf{x}$ to its epipolar line $\mathbf{x}$.

- This map can be considered as projective map. Its matrix is called the fundamental matrix of a pair (P, P'), where P and P' are camera matrices (we usually identify camera with its matrix and we say e.g. P is a camera).

Fundamental matrix – Geometric derivation

- For a 3-vector $\mathbf{v}$ we define cross-product by $\mathbf{v}$ as a map $[\mathbf{v}]_\times$ defined by

$$[\mathbf{v}]_\times \mathbf{u} = \mathbf{v} \times \mathbf{u}.$$

- $[\mathbf{v}]_\times$ is a linear map and for $\mathbf{v} = (v_1, v_2, v_3)$ its matrix is

$$[\mathbf{v}]_\times = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

Fundamental matrix – Geometric derivation

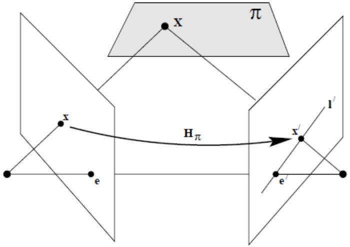

Figure: R. Hartley, A. Zisserman: Multiple View Geometry in Computer Vision

- π – a plane not containing the baseline.
- A ray through $\mathbf{x}$ intersect π at $\mathbf{X}$.
- $\mathbf{x}'$ is image of $\mathbf{X}$ by the second camera.
- This process defines a planar homography H_π between points of image planes.

Fundamental matrix – Geometric derivation

- $\mathbf{l}'$ is defined by $\mathbf{x}'$ and the epipole $\mathbf{e}'$.
- Symbolically

$$\mathbf{l}' = \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_\times \mathbf{x}' = [\mathbf{e}']_\times H_\pi \mathbf{x} = F\mathbf{x}.$$

- The fundamental matrix is then a matrix

$$F = [\mathbf{e}']_\times H_\pi.$$

- This matrix is independent of the chose plane π.

Fundamental matrix – Algebraic derivation

- If P and P' are camera matrices, the image of a ray through $\mathbf{x}$ is

$$X(\lambda) = P^{+}\mathbf{x} + \lambda\mathbf{C}.$$

- Image of a point for $\lambda = 0$ by P' is

$$P'X(0) = P'P^{+}\mathbf{x}.$$

- Epipole $\mathbf{e}'$ in the image plane of the second camera is image by P of first camera centre $\mathbf{e}' = P'\mathbf{C}$
- From this we have

$$\mathbf{l}' = (P'\mathbf{C}) \times (P'P^{+}\mathbf{x}) = [\mathbf{e}']_{\times}(P'P^{+})\mathbf{x}$$

- This shows directly that fundamental matrix $F = [\mathbf{e}']_{\times}(P'P^{+})$ does not depend on the plane π from the geometric derivation.

Fundamental matrix

- For a pair of associated points $\mathbf{x} \leftrightarrow \mathbf{x}'$, the camera matrix F fulfils

$$\mathbf{x}'^T F \mathbf{x} = \mathbf{x}'^T \mathbf{l}' = 0.$$

- This allows finding the fundamental matrix from enough number of pairs $\mathbf{x} \leftrightarrow \mathbf{x}'$.

Fundamental matrix – Properties

- If F is the fundamental matrix of a pair (P, P'), the F^T is the fundamental matrix of the pair (P', P).
- The associated epipolar line $\mathbf{l}'$ to a point $\mathbf{x}$ in the image plane of the first camera is $\mathbf{l}' = F\mathbf{x}$.
- For any $\mathbf{x} \neq \mathbf{e}$ the line $\mathbf{l}' = F\mathbf{x}$ contains epipole $\mathbf{e}'$.
- F has seven degree of freedom.
- F represents a projective map from points to lines.

Camera matrices from fundamental matrix

- If (P, P') is a pair of camera matrices and H is 4×4 projective transformation matrix, then fundamental matrices for pairs (P, P') and $(PH, P'H)$ are the same.
- F determines camera matrices up to a homography.
- Fundamental matrix associated to the pair $P = [I|\mathbf{0}]$ a $P' = [M, \mathbf{m}]$ is $[\mathbf{m}]_\times M$.
- If (P, P') and $(\widetilde{P}, \widetilde{P}')$ have the same fundamental matrix, then there exists a matrix H representing projective transformation such that $\widetilde{P} = PH$ and $\widetilde{P}' = P'H$.

Camera matrices from fundamental matrix

Let F be a fundamental matrix and S an antisymmetric matrix. Let $P = [I|\mathbf{0}]$ and $P' = [SF, \mathbf{e}']$ with $\mathbf{e}'^T F = 0$ an epipole and P' has rank 3. Then F is the fundamental matrix of the pair (P, P').

Essential matrix

- Special case of fundamental matrix for pair of normalized cameras.
- If we know the calibration matrix K of a camera $P = K[R|\mathbf{t}]$ we can define for image of a point $\mathbf{x} = P\mathbf{X}$ its normalized coordinates as

$$\hat{\mathbf{x}} = K^{-1}\mathbf{x}.$$

I.e.

$$\hat{\mathbf{x}} = [R|\mathbf{t}]\mathbf{X}.$$

- Camera with matrix $K^{-1}P = [R|\mathbf{t}]$ is called normalized camera.

Essential matrix

- Essential matrix is the fundamental matrix for a pair $P = [I, \mathbf{0}]$ a $P' = [R, \mathbf{t}]$ of normalized cameras.
- Essential matrix is defined as

$$E = [\mathbf{t}]_\times R = R[R^T \mathbf{t}]_\times .$$

- The equation of essential matrix is

$$\hat{\mathbf{x}}'^T E \hat{\mathbf{x}} = 0.$$

- Relation between fundamental and essential matrix is given by

$$F = K'^T E K$$

where K' and K are calibration matrices of the cameras.

Camera matrices from essential matrix

- SVD of essential matrix $E = U\mathrm{diag}(1,1,0)V^T$
- Let $W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ a $Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
- Then $E = SR$ with S an antisymmetric matrix and R an orthogonal matrix given by

$$S = UZU^T \qquad\qquad R = UWV^T \text{ or } UW^TV^T.$$

- This gives 4 possibilities for camera P' such that (P, P') with $P = [I|\mathbf{0}]$ has E as the essential matrix

$$P' = [UWV^T| \pm \mathbf{u}_3], \text{ or}$$
$$P' = [UW^TV^T| \pm \mathbf{u}_3],$$

with $\mathbf{u}_3$ as the last column of the matrix U.