Affine and metric reconstruction

Affine reconstruction

- Line at infinity $\mathbf{l}_\infty = (0,0,1)^T$ is transformed by affine transformation onto line at infinity

$$H_A^{-T}\mathbf{l}_\infty = \begin{pmatrix} A^{-T} & \mathbf{0} \\ -\mathbf{t}^T A^{-T} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty .$$

Affine reconstruction

- If a projective transformation $H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$ transforms line at infinity onto line at infinity, then the transformation H is affine, because then a point in infinity $(x, 0, 0)$ maps to a point $(*, *, 0)$, i.e. $h_{31} = 0$ and similarly for $(0, y, 0)$, $h_{32} = 0$.

Affine reconstruction

To reconstruct the affine structure in a given image:

- Find the image of line at infinity – this is a line in the image, which can be outside of viewpoint.
- Find projective transformation, witch transforms the line from previous point onto line at infinity.
- For line $\mathbf{l}$ represented by $\mathbf{l} = (l_1, l_2, l_3)^T$ the trasformation is

$$H = H_A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{pmatrix},$$

with H_A an arbitrary affine transformation.

Affine reconstruction

To find an image of the line at infinity:

- Find a pair of lines in the image that represent parallel lines in reality.
- Their intersection is a point which represents image of an ideal point.
- Two such pairs that give two different intersections define a line, witch represents the image of the line at infinity.

Affine reconstruction

Figure: Original image without any reconstruction. The image of the line at infinity can be obtained using intersections of two pairs of parallel lines, e.g. pairs of opposite sides of the tiles.

Affine reconstruction

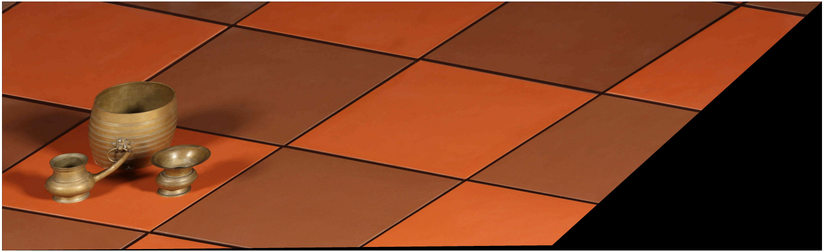

Figure: Image after affine reconstruction. The lines that should be parallel are parallel

Metric reconstruction

Object of interest for metric reconstruction is a pair of 'circular points'. These are points obtained as intersection of a circle and the line at infinity. In Euclidean case:

- Homogeneous equation of a circle is

$$(x - aw)^2 + (y - bw)^2 = r^2 w^2.$$

- Equation of line in infinity is $w = 0$.
- The circular points are then the solution of

$$x^2 + y^2 = 0.$$

- Two complex solutions in homogeneous coordinates are

$$\mathbf{I} = (1, i, 0) \qquad \mathbf{J} = (1, -i, 0).$$

Metric reconstruction

- Circular points are similarity invariants. If H_S is similarity matrix, then

$$H_S \mathbf{I} == \mathbf{I}, \qquad H_S \mathbf{J} == \mathbf{J}$$

- It can be shown that any projective transformation for which the circular points are invariant is similarity.
- To perform a metric reconstruction one needs to find images of circular points.

Metric reconstruction

To simplify the process of metric reconstruction (i.e. finding the circular points) one can define an object called 'conic dual to circular points' which codes the circular points in it and has an advantage to being real

- Conic dual to circular points is defined as matrix

$$C_\infty^* = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T.$$

- In Euclidean case

$$C_\infty^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- C_∞^* is similarity invariant

$$C_\infty^{*\prime} = H_S C_\infty^* H_S^T = C_\infty^*$$

Metric reconstruction

- Knowledge of C_∞^* admits definition of projective invariant angle of two lines $\mathbf{l}$ and $\mathbf{m}$.

$$\cos\theta = \frac{\mathbf{l}^T C_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^T C_\infty^* \mathbf{l})(\mathbf{m}^T C_\infty^* \mathbf{m})}}$$

- Two lines $\mathbf{l}$ and $\mathbf{m}$ are perpendicular iff $\mathbf{l}^T C_\infty^* \mathbf{m} = 0$.

Metric reconstruction

- Every projective transformation H can be decomposed as

$$H = H_P H_A H_S = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{v}^T & v \end{pmatrix} \begin{pmatrix} K & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} sR & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

- The image of (Euclidean) C^*_∞ by a transformation H is

$$\begin{aligned} C^{*\prime}_\infty &= (H_P H_A H_S) C^*_\infty (H_P H_A H_S)^T = (H_P H_A) C^*_\infty (H_P H_A)^T \\ &= \begin{pmatrix} KK^T & KK^T\mathbf{v} \\ \mathbf{v}^T KK^T & \mathbf{v}^T KK^T \mathbf{v} \end{pmatrix} \end{aligned}$$

Metric reconstruction

- For affine rectified picture the image of conic dual to circular points has matrix

$$C_\infty^* = \begin{pmatrix} s_1 & s_2 & 0 \\ s_2 & s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

  where K is upper triangular and R is orthogonal.
- If $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$ are perpendicular, the perpendicularity condition for conic dual gives an equation

$$(l_1 m_1, l_1 m_2 + l_2 m_1, l_2 m_2) \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0.$$

Metric reconstruction

- Two pairs of perpendicular lines yields a system of two equations.
- Its solution defines C^*_∞ up to a scale.
- To find K from H_A we can use Cholesky decomposition of positive definite symmetric matrix $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$

Metric reconstruction–Cholesky decomposition

- S is positive definite symmetric matrix
- By singular value decomposition $S = UDU^T$ with D a diagonal matrix and U an orthogonal matrix.
- Write $D = EE^T$, i.e. D is the square root of E.
- Then $S = (UE)(UE)^T$.
- $(UE) = RQ$ from RQ-decomposition. R is upper triangle and Q is orthogonal.
- $S = (RQ)(RQ)^T = RR^T$. And we set $K = R$.

Metric reconstructon

Figure: Image after metric rectification. Notice the tiles are squares and ellipses became circles. Also notice distortion of the metallic objects. It is caused by assumption that they are planar in reality (they are not!).