Multiple view geometry

Projective geometry – an overview

Projective space

- World space is modeled as Euclidean space $\mathbb{R}^3$.
- Clasical camera model projects world space onto a plane modeled as $\mathbb{R}^2$.
- A affine coordinate system in $\mathbb{R}^2$ admits a description of points as ordered pairs of real numbers (a, b) – affine coordinates of points. Similarly, for $\mathbb{R}^3$ coordinates are triples.
- Augmented affine coordinates are affine coordinates of the form $(a, b, 1)$.

Projective space

- Define equivalence relation on triples (a, b, c)

$$(a, b, 1) \sim (ka, kb, k),\ \ k \neq 0.$$

- This defines homogeneous coordinates on $\mathbb{R}^2$. A point $X \in \mathbb{R}^2$ has homogeneous coordinates (a, b, c) iff X has affine coordinates $(\frac{a}{c}, \frac{b}{c})$
- There are tripples $(x, y, 0) \neq (0, 0, 0)$.These represent, as homogeneous coordinates, so called ideal points (or points in infinity).
- The set containing homogeneous coordinates of points of $\mathbb{R}^2$ and homogeneous coordinates of ideal points is projective space $\mathbb{P}^2$.

Lines in $\mathbb{P}^2$

- Equation of line in $\mathbb{R}^2$ is

$$ax + by + c = 0.$$

- Homogenization $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$,

$$ax_1 + bx_2 + cx_3 = 0$$

  is equation of the line in $\mathbb{P}^2$.
- The line can be then defined as triple $\mathbf{l} = (a, b, c)^T$, with (a, b, c) a homogeneous triple, i.e. (a, b, c) and (ka, kb, kc), $k \neq 0$ define the same line.
- Homogeneous point $\mathbf{x}$ is point of line $\mathbf{l}$ iff $\mathbf{l}^T\mathbf{x} = 0$.

Lines in $\mathbb{P}^2$

- Intersection of two lines $\mathbf{l}$ and $\mathbf{m}$ is a the point $\mathbf{x} = \mathbf{l} \times \mathbf{m}$.
- Similarly, the line defined by two points $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{l} = \mathbf{x} \times \mathbf{y}$.
- The line can be then defined as triple $\mathbf{l} = (a, b, c)^T$, with (a, b, c) a homogeneous triple, i.e. (a, b, c) and (ka, kb, kc), $k \neq 0$ define the same line.
- Homogeneous point $\mathbf{x}$ is point of line $\mathbf{l}$ iff $\mathbf{l}^T\mathbf{x} = 0$.

Conics in $\mathbb{P}^2$

- The general equation of conic in $\mathbb{R}^2$ is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

- Homogenization gives

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0.$$

- five points (x_i, y_i), $i = 1, \ldots 5$ define a conic.

Conics in $\mathbb{P}^2$

- Homogeneous conic equation is a quadratic form, i.e. it is represented by symmetric matrix

$$\begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

- Point $\mathbf{x}$ is point of conic C iff $\mathbf{x}^T C \mathbf{x} = 0$
- five points (x_i, y_i), $i = 1, \ldots 5$ define a conic.

Conics in $\mathbb{P}^2$

- Tangent line $\mathbf{l}$ to conic C at point $\mathbf{x} \in C$ is $\mathbf{l} = C\mathbf{x}$.

Proof: $\mathbf{x} \in \mathbf{l} = C\mathbf{x}$ because $\mathbf{l}^T x = \mathbf{x}^T C^T \mathbf{x} = \mathbf{x}^T C\mathbf{x} = 0$. If this is the only point, then $\mathbf{l}$ is tangent. If $\mathbf{y} \in C$ is another point of line at $\mathbf{x}$ then in can be proved that $\mathbf{l} \subset C$.

Dual conic

- If matrix C represent a conic, then adjoint C^* represent a conic called dual conic to C.
- If C is regular then up to scale $C^* = C^{-1}$.
- Conic C defines equations on points, dual conic C^* defines equations on lines.
- Derivation of dual conic for regular conic C: Tangent at $\mathbf{x}$ is $\mathbf{l} = C\mathbf{x}$, i.e. $\mathbf{x} = C^{-1}\mathbf{l}$. Then

$$0 = \mathbf{x}^T C \mathbf{x} = \mathbf{l}^T C^{-1} C C^{-1} \mathbf{l} = \mathbf{l} C^{-1} \mathbf{l}.$$

Projective transformations

- Projective transformation (or projectivity) is an invertible mapping $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that if points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ lie on the same line, then $h(\mathbf{x}_1)$, $h(\mathbf{x}_2)$, $h(\mathbf{x}_3)$ lie on the same line.
- A projectivity can be represented by 3×3 regular matrix

Projective transformations of lines

- If H is projective transformation of points, $\mathbf{x}' = H\mathbf{x}$, and if $\mathbf{x} \in \mathbf{l}$, then $0 = \mathbf{l}^T\mathbf{x} = \mathbf{l}^T H^{-1} H\mathbf{x} = (H^{-T}\mathbf{l})^T\mathbf{x}'$, i.e. $\mathbf{x}'$ are points of a line $H^{-T}\mathbf{l}$.
- This means that if points are transformed by H, lines are transformed by H^{-T}.

Projective transformations of conics

- If H is projective transformation of points, $\mathbf{x}' = H\mathbf{x}$ and C is a conic then

$$\mathbf{x}^T C\mathbf{x} = \mathbf{x}'^T H^T C H\mathbf{x} = \mathbf{x}'^T C'\mathbf{x}'$$

- This means that projective transformation H of points transforms conic C to conic $H^T C H$.
- Similarly, projective transformation H of points transforms dual conic C^* to conic $H C^* H^T$.

Types of transformations

Isometry:

- Matrix

$$H_E = \begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

- $R \in O(2)$, $\mathbf{t} \in \mathbb{R}^2$.
- 3 degrees of freedom
- Invariants: Length, angle, area

Types of transformations

Similarity:

- Matrix

$$H_S = \begin{pmatrix} sR & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

- $R \in O(2)$, $\mathbf{t} \in \mathbb{R}^2$, $s \in \mathbb{R}$.
- 4 degrees of freedom
- Invariants: Ratios of lengths, angle, ratios of areas

Types of transformations

Affine transform:

- Matrix

$$H_A = \begin{pmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

- $R \in GL(2)$, $\mathbf{t} \in \mathbb{R}^2$.
- 6 degrees of freedom
- Invariants: Parallelism, Ratios of lengths of parallel line segments.

Types of transformations

Projective transform:

- Matrix

$$H_A = \begin{pmatrix} M & \mathbf{t} \\ \mathbf{v}^T & v \end{pmatrix}$$

- M is 2×2 matrix, $\mathbf{t}, \mathbf{v} \in \mathbb{R}^2$, $v \in \mathbb{R}$.
- 8 degrees of freedom
- Invariants: Cross ratio.