Image preprocessing

Frequency analysis and filtering II

Sampling

Sampling is restriction of continuous domain of signal $f(x)$ onto discrete subset, e.g.

- $f : \mathbb{R} \to \mathbb{R}$ – signal,
- Δx – sampling period
- Sampled signal $f_s(n) = f(n\Delta x)$, $n \in \mathbb{Z}$.

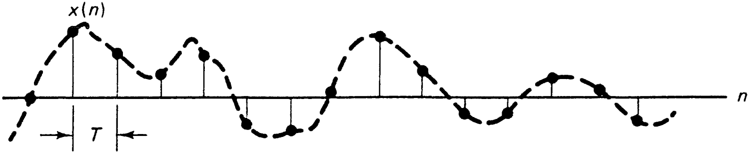


Figure: Here $T = \Delta x$, $x(n) = f_s(n)$

Sampling

More generally

$$f_s(n) = f(q(n))$$

- $q : \mathbb{U} \rightarrow \mathbb{R}$,
- U is discrete subset of $\mathbb{R}$
- In the previous example $q(n) = n\Delta x$.

Models of signal sampling

Dirac δ 'function'

- $\delta : \mathbb{R} \to \mathbb{R}$
- $\delta(x) = 0$ for all $x \neq 0$.
- $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Represents impulse with finite energy.
It is not a function in the strict sense. It is so called 'generalized funcion' or 'distribution'.

- Sifting property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = a.$$

Model of signal sampling

Define impulse train

$$r(x) = \sum_{n \in \mathbb{Z}} \delta(x - n\Delta x)$$

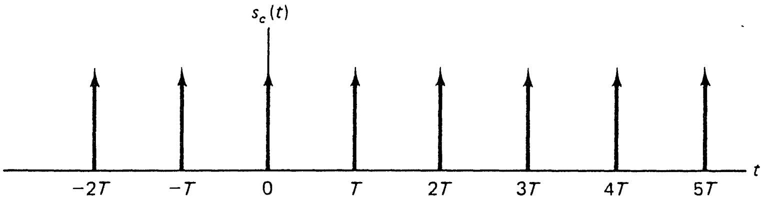


Figure: Impulse train $T = \Delta x$, Jae S. Lim, Alan V. Oppenheim:Advanced topics in signal processing.

Model of signal sampling

Modulate (multiply) signal by impulse train

$$f(x)r(x) = f(x) \sum_{n \in \mathbb{Z}} \delta(x - n\Delta x) \tag{1}$$

Integrate modulated signal in neighborhoods of samples

$$f_s(n) = \lim_{\epsilon \to 0} \int_{n\Delta x - \epsilon}^{n\Delta x + \epsilon} f(x)r(x)dx.$$

Modulated system in frequency domain

Convolution theorem for fourier transform

$$\mathcal{F}[f \cdot r] = F * R$$

- $\mathcal{F}$ – Fourier transform operator
- $F = \mathcal{F}(f)$, $R = \mathcal{F}(r)$

Modulated system in frequency domain

What is R?

- Impulse train $r(x)$ is periodic with period Δx
- We can find Fourier series for periodic function r

$$r(x) = \sum_{n \in \mathbb{Z}} c_n e^{-2\pi i \frac{n}{\Delta x} x}$$

- Coefficients of Fourier series of impulse train are $c_n = \frac{1}{\Delta x}$

Modulated system in frequency domain

We apply Fourier transform on

$$r(x) = \frac{1}{\Delta x} \sum_{n \in \mathbb{Z}} e^{-2\pi i \frac{n}{\Delta x} x}$$

and get

$$R(\xi) = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(\xi - \frac{n}{\Delta x})$$

Conclusion

- Fourier transform of impulse train in space domain is impulse train in frequency domain.
- Shorter impulse train period in space domain gives longer impulse train period in frequency domain and v.v.

Modulated system in frequency domain

Convolution with signal

$$(F * R)(\xi) = \int_{-\infty}^{\infty} F(\tau)R(\xi - \tau)d\tau = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} F(\xi - \frac{n}{\Delta x})$$

Graphically for band limited signal

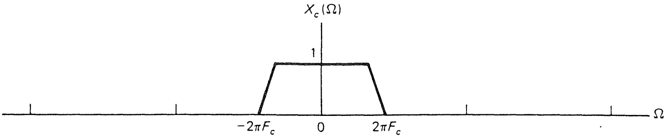


Figure: $-2\pi F_c$ is cutoff frequency, Jae S. Lim, Alan V. Oppenheim:Advanced topics in signal processing.

Modulated system in frequency domain

Impulse train in frequency domain

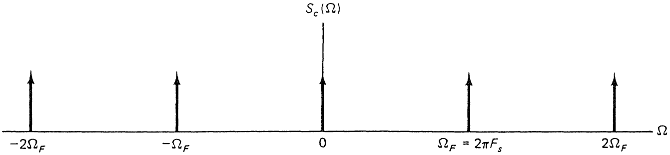
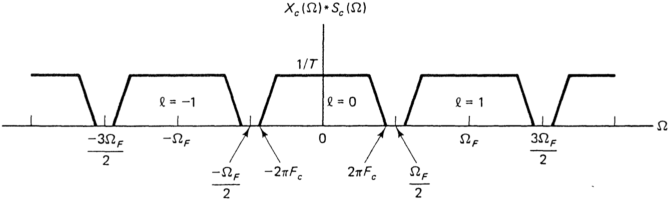


Figure: Jae S. Lim, Alan V. Oppenheim:Advanced topics in signal processing.

and resulting convolution



Modulated system in frequency domain

- Fourier transform of digital signal is the sum of frequency-shifted and scaled version of the Fourier transform of the continuous signal.

Modulated system in frequency domain – aliasing

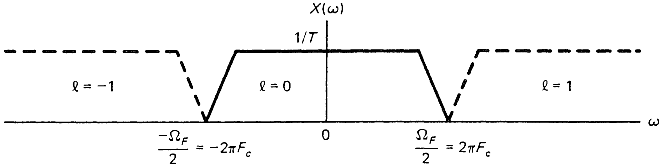


Figure: Result of sampling with sampling frequency equal to double of cut-off frequency

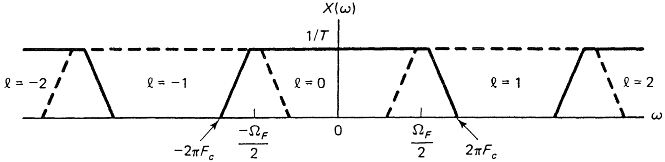


Figure: Result of sampling with sampling frequency less than double of cut-off frequency – aliasing occurs.

Modulated system in frequency domain – aliasing

- In aliased case, signal cannot be reconstructed back in space domain from frequency domain.

Theorem

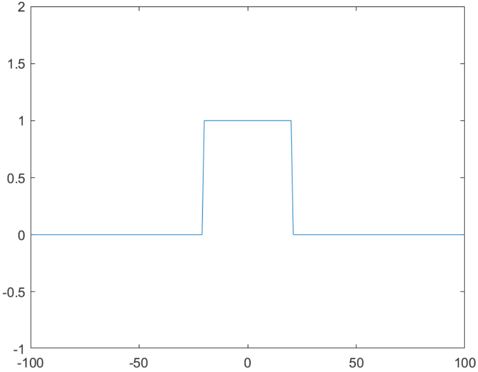
If continuous signal $f(x)$ has a band limited spectrum, i.e. $|F(\xi)| = 0$ for $\xi \geq F_c$, then f can be uniquely reconstructed from equally spaced samples $f_s(n) = f(nT)$ if sampling frequency is at least two times the highest frequency in the signal.

Filtering in frequency domain

- Transform signal from space to frequency domain $f \rightsquigarrow F$
- Multiply the transformed signal by function G called filter. $H = F \times G$. The filter G is zero at those frequencies which need to be suppressed.
- Transform resulting spectrum back to space domain $H \rightsquigarrow h$

Useful filters

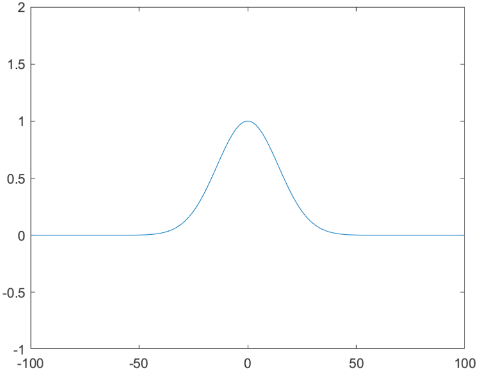
- Ideal filter



- Low-pass filter.
- Suppresses all frequencies which are above cut-off frequency of the filter.

Useful filters

- Gaussian filter



- Low-pass filter.

High-pass filters construction

If G is a low-pass filter, then high-pass filter H can be constructed as

$$H = 1 - G$$

Discrete Fourier transform

Describes spectral properties of discrete signal. It is similar to continuous Fourier transform.

- for $f(n)$, $n = 0, \dots, N-1$ a discrete signal, discrete Fourier transform is defined as

$$F(k) = \sum_{n=0}^{N-1} f(n) e^{-2\pi i \frac{nk}{N}}.$$

and inverse Fourier transform is

$$f(n) = \sum_{k=0}^{N-1} F(k) e^{2\pi i \frac{nk}{N}}.$$

Discrete Fourier transform in 2D

- for $f(m,n)$, $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$ a discrete signal, discrete Fourier transform is defined as

$$F(u,v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

$$u = 0, 1, \ldots, M-1, \quad v = 0, 1, \ldots, N-1,$$

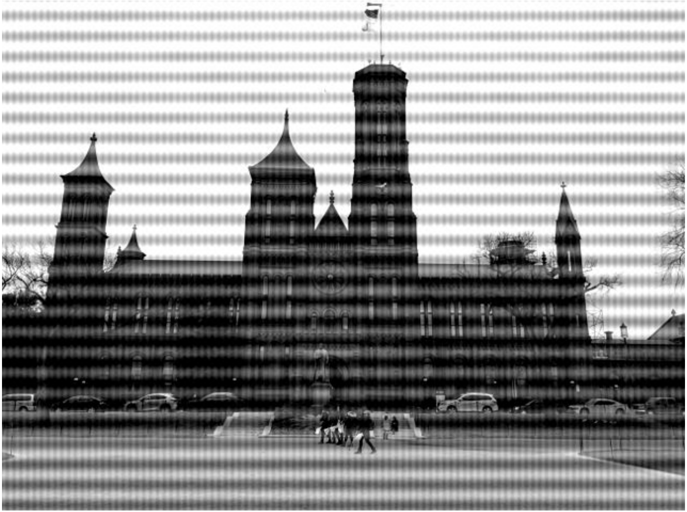
a inverz

$$f(m,n) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

$$m = 0, 1, \ldots, M-1, \quad n = 0, 1, \ldots, N-1,$$

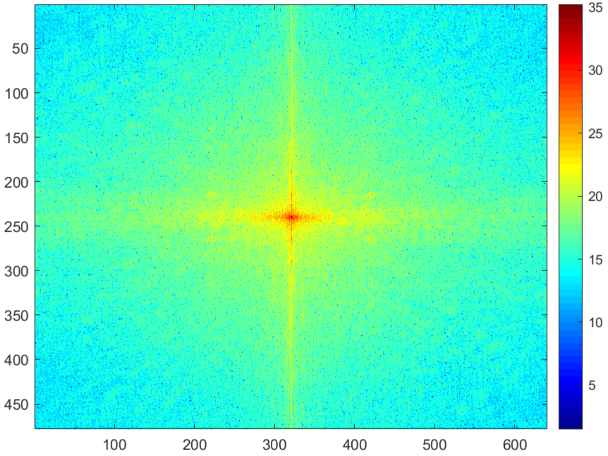
Example of 2D DFT application – Periodic noise

We have image distorted by periodic noise



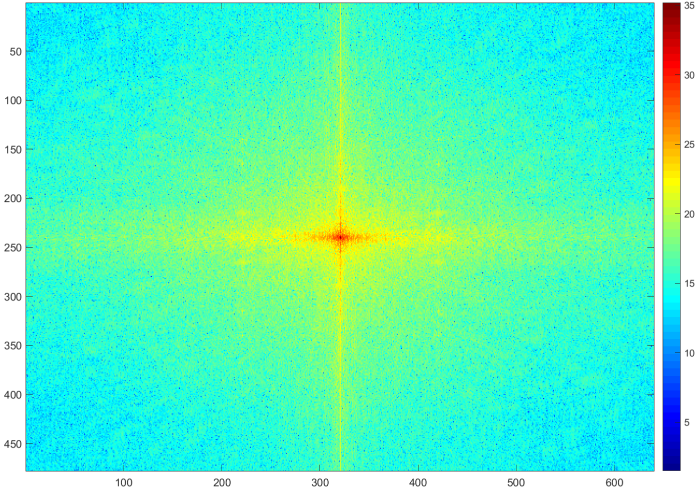
Example of 2D DFT application – Periodic noise

Spectrum of previous image obtained using 2D DFT. Notice the red pixels (peaks in the spectrum) – hard to see.



Example of 2D DFT application – Periodic noise

Spectrum with smoothed out peaks



Example of 2D DFT application – Periodic noise

Image with removed noise