Image preprocessing

Frequency analysis and filtering

Fourier series

Jean Baptiste Joseph Fourier (1768–1830)

- 1822—The Analytical Theory of Heat
- Any function of a variable can be expanded in a series of sines of multiples of the variable
- Was not accepted at his time

Fourier series

- For T-periodic function f, which is integrable on interval of length T

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)$$

- Functions $\sin$ and $\cos$ are orthogonal, i.e.

$$\int_T \sin\left(\frac{2\pi n}{T}x\right) \cos\left(\frac{2\pi m}{T}x\right) dx = 0$$

Fourier series

$$\int_T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

similarly for $\sin$

$$\int_T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

This admits computation of coefficients a_n and b_n.

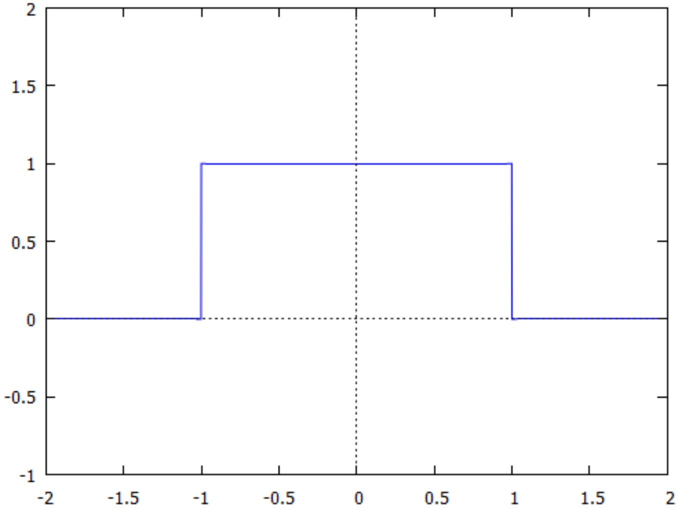
Fourier coefficients

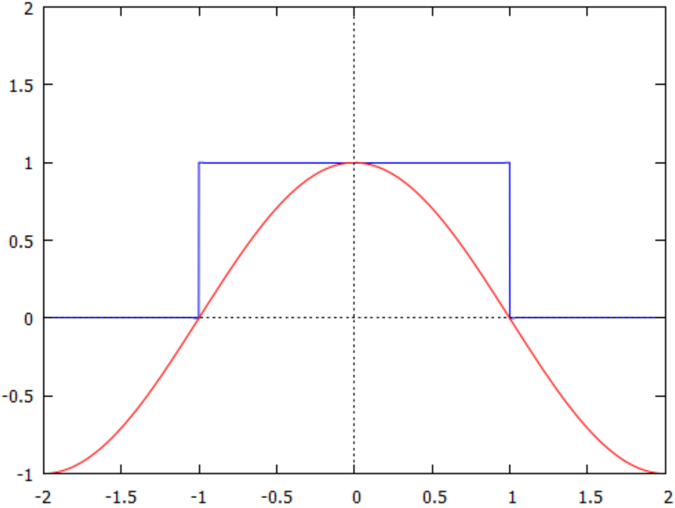
$$a_n = \frac{2}{T}\int_T f(x)\cos\left(\frac{2\pi n}{T}x\right)dx,$$
$$b_n = \frac{2}{T}\int_T f(x)\sin\left(\frac{2\pi n}{T}x\right)dx,$$

Geometric viewpoint

- $a_0/2$ represents mean value of function f on a given interval
- a_n and b_n represent how much functions $\sin$ and $\cos$ of given frequencies correlate with function f.

Geometric viewpoint

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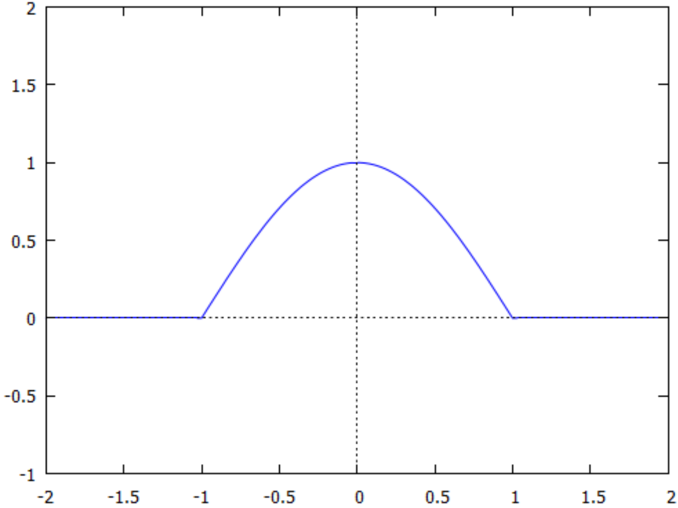

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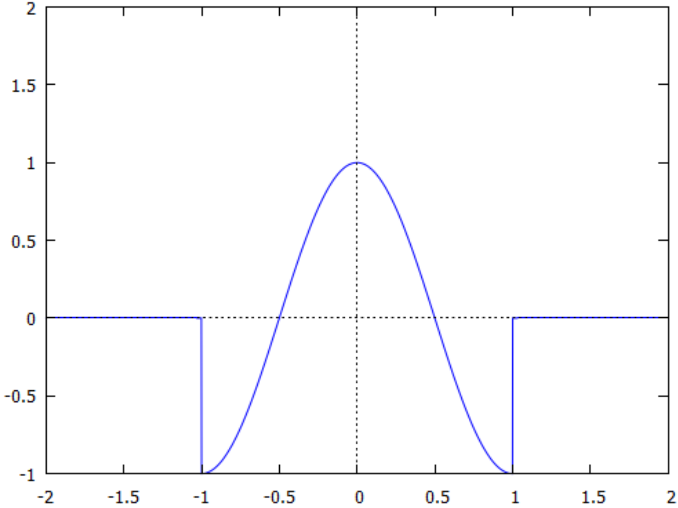

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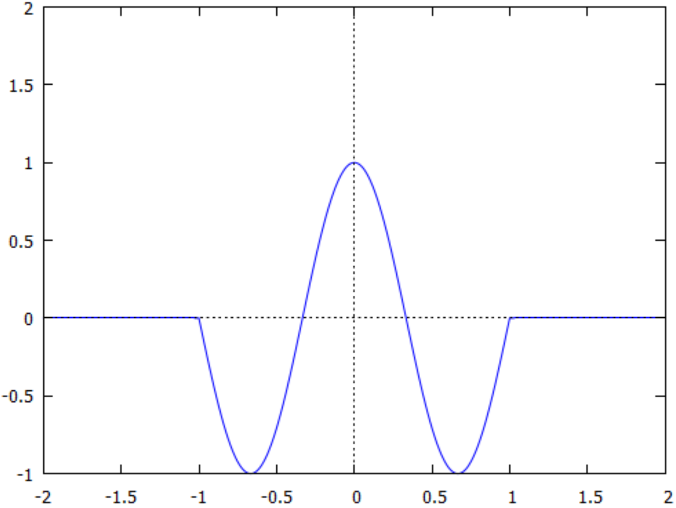
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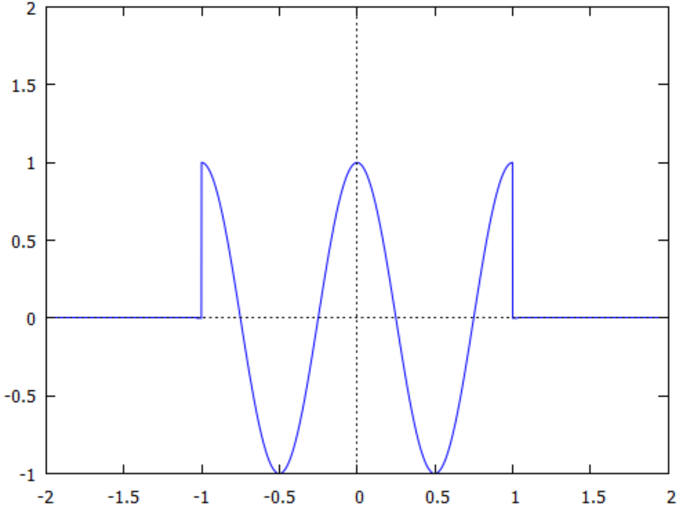
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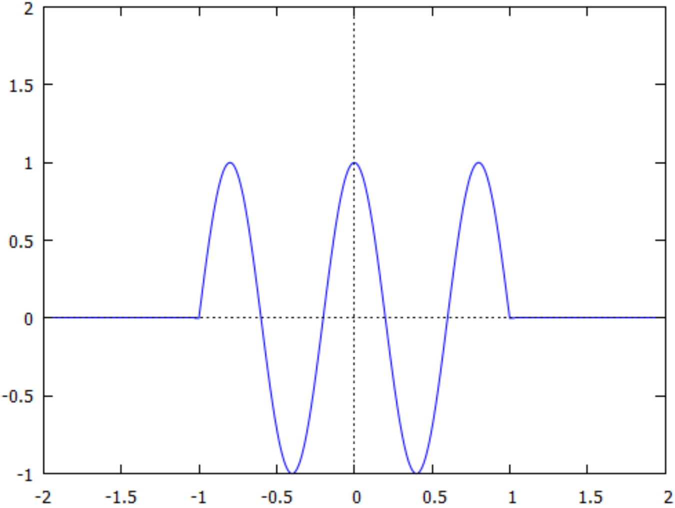
2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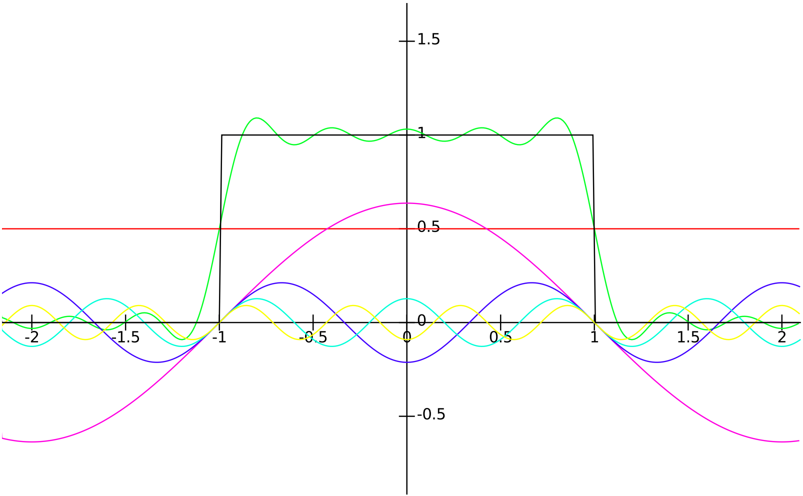
2



1.5
1
0.5
0
-0.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Complex version

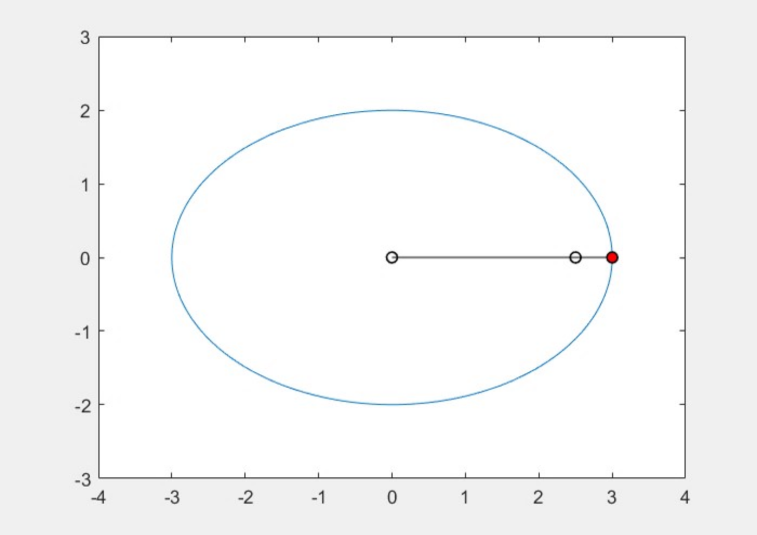
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{T} ix}$$

- Suggested by Euler's formula

$$e^{ix} = \cos x + i \sin x$$

- c_n is in general a complex number.
-

$$c_n = \frac{1}{T} \int_T f(x) e^{-\frac{2\pi n}{T} ix}$$



3
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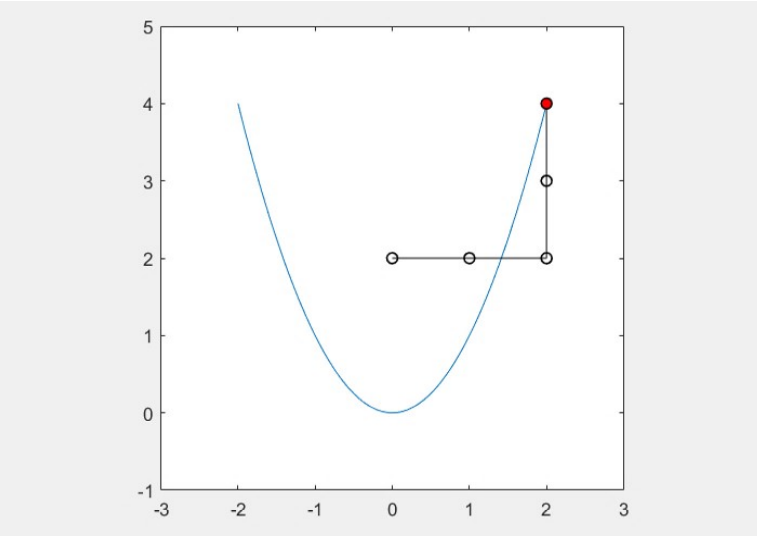
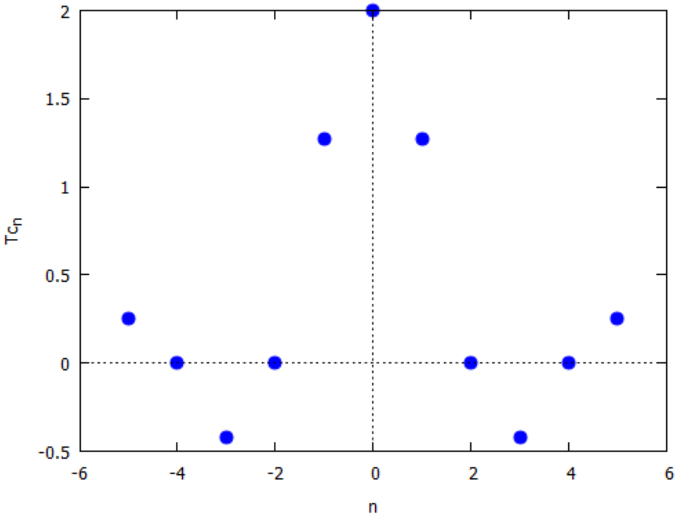
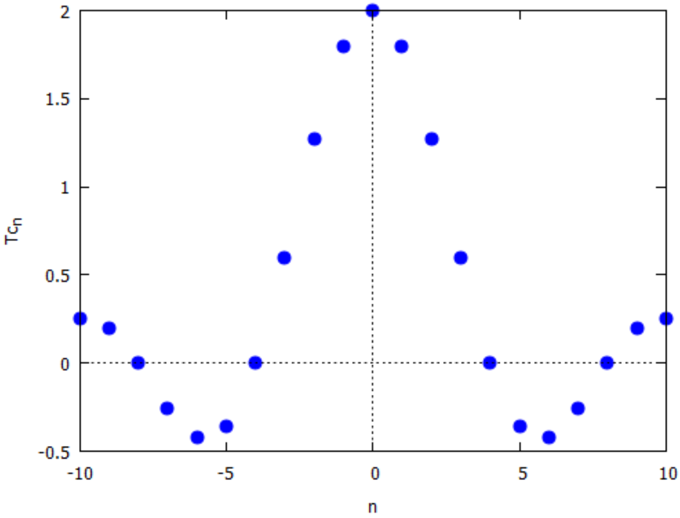
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4


5
4
3
2
1
0
-1
-3
-2
-1
0
1
2
3

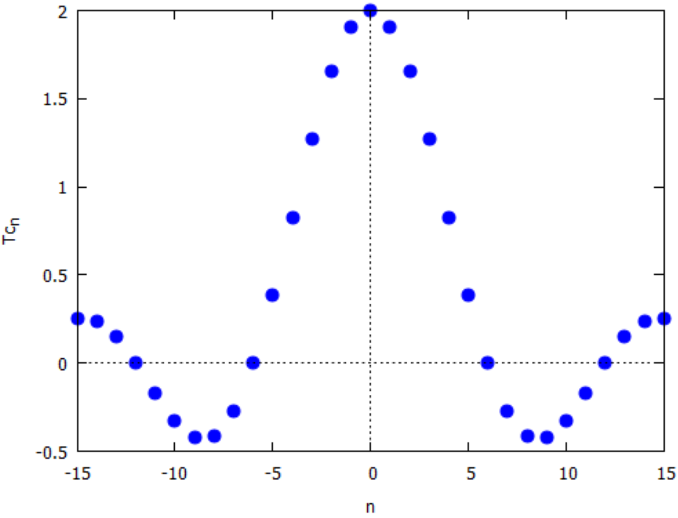
Coefficient plot, $T = 4$



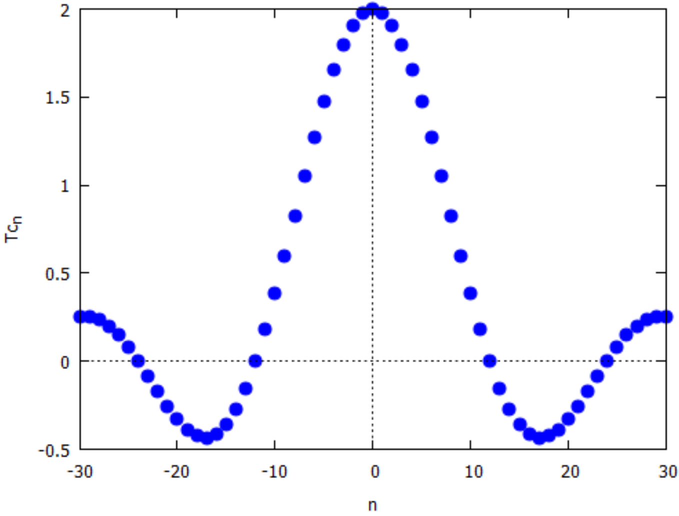
Coefficient plot, $T = 8$



Coefficient plot, $T = 12$



Coefficient plot, $T = 24$



Fourier transform

If $T \to \infty$, for particular frequency $\xi = \frac{n}{T}$ value of integral

$$Tc_n = \int_T f(x) e^{-\frac{2\pi n}{T} ix} dx$$

remains unchanged. We then put

$$\mathcal{F}\{f(x)\} = F(\xi) = Tc_n = \int_{-\infty}^{\infty} f(x) e^{-2\pi \xi ix} dx.$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}\{F(x)\} = f(x) = \int_{-\infty}^{\infty} F(x) e^{2\pi \xi ix} dx.$$

Useful properties

- Linearity

$$\mathcal{F}\{af(x) + bf(y)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

- Duality

$$\mathcal{F}\{F(x)\} = f(-\xi)$$

- Product

Convolution

$$(f * h)(x) \equiv \int_{-\infty}^{\infty} f(x)h(x-\tau)d\tau = \int_{-\infty}^{\infty} f(x-\tau)h(x)d\tau$$

- Expresses amount of overlap of one function $f(x)$ as it is shifted over another function $h(t)$.

Convolution – properties

$$f * h = h * f$$
$$f * (g * h) = (f * g) * h$$
$$f * (g + h) = f * g + f * h$$
$$a(f * g) = (af) * g = f * (ag)$$
$$\frac{d}{dx}(f * h) = \frac{df}{dx} * h = f * \frac{dh}{dx}$$

Application of FT

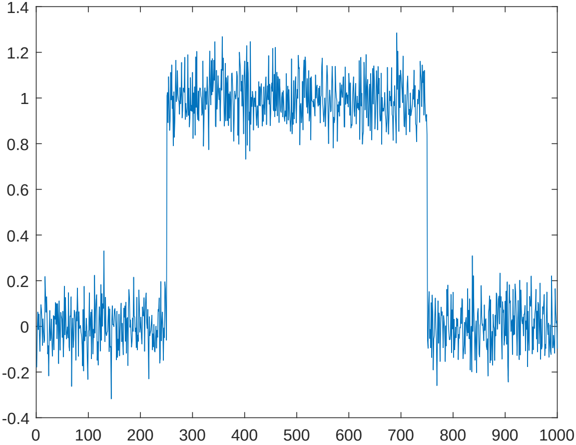


Figure: Rectangular signal with noise

Application of FT

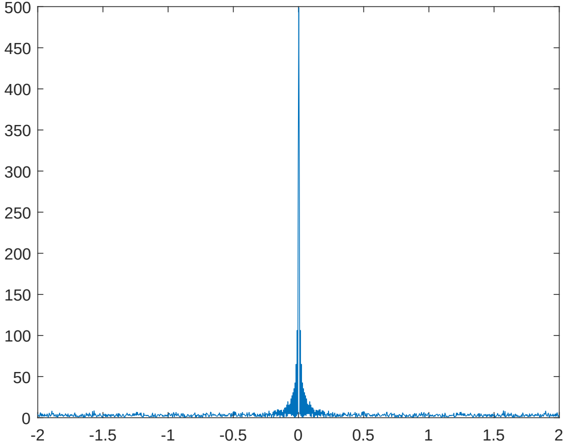


Figure: Frequency spectrum of previous signal

Application of FT

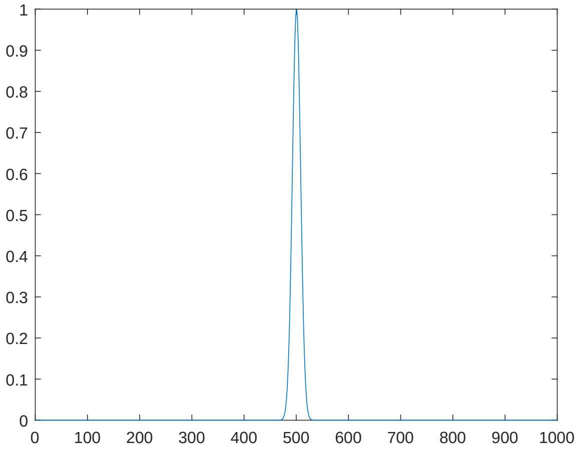


Figure: Low-pass filter – gaussian $e^{-a\xi^2}$

Application of FT

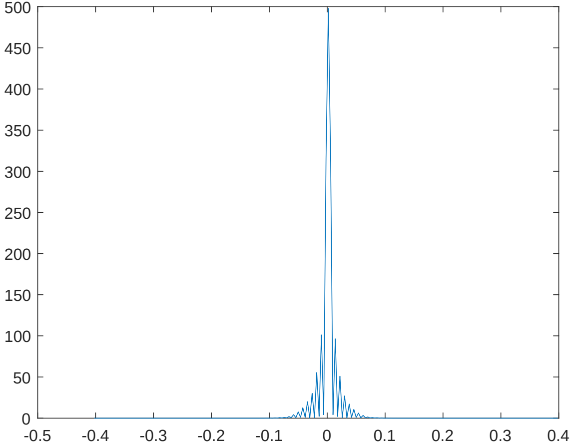


Figure: High frequencies suppressed

Application of FT

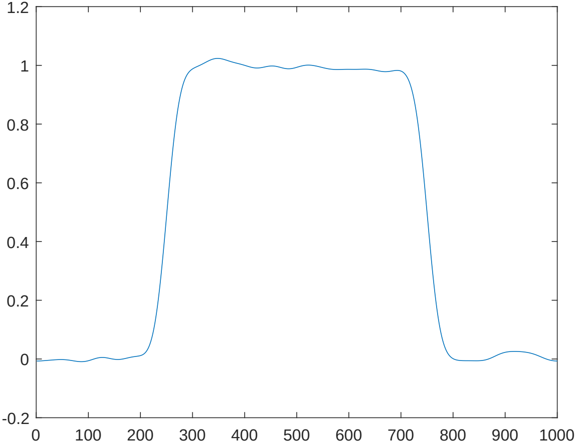


Figure: Rectangular function with suppressed noise

Back to Fourier

- Product

$$\mathcal{F}\{f(x)g(x)\} = F(\xi) * G(\xi)$$

- Convolution

$$\mathcal{F}\{(f * g)(x)\} = F(\xi)G(\xi)$$

Discrete Fourier transform (DFT)

- Computers deal with discrete signal $f(n),\ n = 0, \dots N-1$.
- DFT is defined as

$$F(k) = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-2\pi i \frac{nk}{N}}.$$

- Inverse DFT is defined as

$$f(n) = \sum_{k=0}^{N-1} F(k) e^{2\pi i \frac{nk}{N}}.$$

- Spectrum is periodic with period N.

Adding a dimension

- In 2D the Fourier transform can be generalized as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(xu+yv)}dx\,dy,$$

- and inverse transform is

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{2\pi i(xu+yv)}du\,dv,$$

Convolution theorem

$$\mathcal{F}\{(f * h)(x,y)\} = F(u,v)H(u,v)$$
$$\mathcal{F}\{(f(x,y)h(x,y)\} = (F * H)(u,v)$$

Discrete version

- DFT in 2D

$$F(u,v) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-2\pi i\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

$$u = 0,1,\ldots,M-1, \quad v = 0,1,\ldots,N-1,$$

- and its inverse

$$f(m,n) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{2\pi i\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

$$m = 0,1,\ldots,M-1, \quad n = 0,1,\ldots,N-1,$$

Different spectra

- Complex spectrum – hard to visualise

$$F(\xi) = \Re(F(\xi)) + i\Im(F(\xi))$$

Suitable for visualisation

- Amplitude spectrum

$$|F(\xi)| = \sqrt{\Re(F(\xi))^2 + \Im(F(\xi))^2}$$

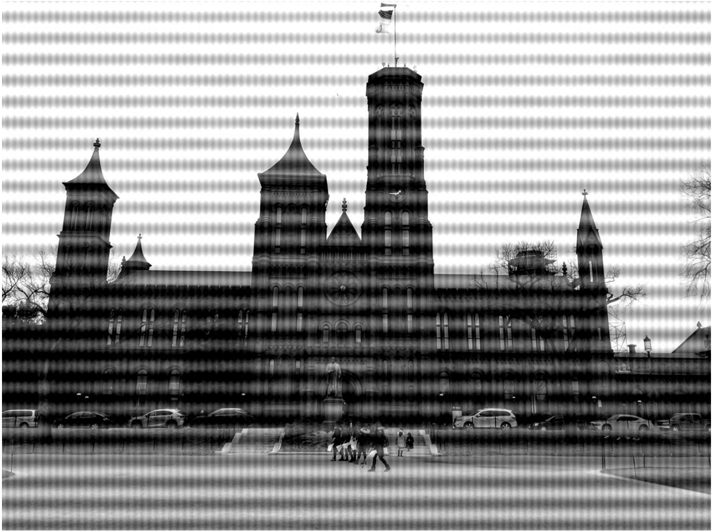
- Phase spectrum

$$\Phi(\xi) = \arctan\left(\frac{\Im(F(\xi))}{\Re(F(\xi))}\right)$$

- Power spectrum

$$P(\xi) = |F(\xi)|^2$$

Image with periodic noise



Amplitude spectrum